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# Gluon Radiation and Energy Losses in Top Quark Production

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## Abstract

The emission of energetic gluons in  $t\bar{t}$  production in  $e^+e^-$  annihilation can have important experimental consequences, in particular on top quark mass measurements. We present compact, analytical expressions for the gluon energy distribution and its average value at first order in QCD perturbation theory. Our results are valid for arbitrary masses, collision energies and production currents. We pay particular attention to top quark production near threshold, and show that in certain cases the soft gluon approximation is insufficient to describe the radiation spectrum.

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# 1 Introduction

The discovery of the top quark – one of the basic components of the Standard Model – is one of the most important goals for present and future experiments. Indirect evidence for the existence of top is very strong, see for example [1], and the chances that it will be detected at the Fermilab  $p\bar{p}$  collider in the next few years are very high. Once top is discovered, the next challenge for experiments is to measure its parameters – in particular its mass  $M$  – as precisely as possible.

An unambiguous interpretation of experimental data and the determination of the top quark parameters relies on a clear quantitative understanding of the details of the production process, including the effects of gluon bremsstrahlung at the production stage [2, 3]. To quantify these effects, one can study the average amount of energy lost by the top quark – the simplest infrared-safe characteristic of gluon bremsstrahlung. Such a quantity is of potential practical importance in top mass determinations and, for energies above threshold, can be calculated quite straightforwardly without undue complications from non-perturbative phenomena or top-width effects [2, 3, 4]. The key point here is the large top width,  $\Gamma_t$  [5]. Since we know from experiment that the top mass is greater than about 120 GeV [6], the decay width is greater than the typical hadronic scale,  $\Gamma_t > \mu \sim 1 \text{ fm}^{-1}$ , and so the top quark decays before non-perturbative effects like the formation of  $t$ -flavoured hadrons become important [5, 7].

The bremsstrahlung of gluons off the top quark can have an important impact (“radiation damage”) on various characteristics of the  $t\bar{t}$  final state. Taking  $e^+e^- \rightarrow t\bar{t}$  as an example, it obviously affects the kinematical constraint  $E_t = E_{\text{beam}}$ , over and above the well-known effects of initial radiation.<sup>2</sup> It also influences the form of the  $t\bar{t}$  production vertex and measurements of the spectra of secondary leptons from  $t$ -decay [10, 11].

Since one of the primary goals of a future  $e^+e^-$  collider would be to obtain an accurate measurement of the top mass  $M$ , it is worth considering the impact of gluon radiation on such a measurement. There are two scenarios which have been considered [12]. If the top is not too heavy, it can have sizeable kinetic energy and a mass measurement can be obtained from the invariant mass per hemisphere. In practice, the distributions in the heavy ( $M_H$ ) and light ( $M_L$ ) hemisphere masses can be fitted to a Monte Carlo simulation with  $M$  as a free parameter. However, a gluon radiated from the  $t\bar{t}$  system will distort this measurement. In particular the average difference in the hemisphere masses squared is proportional (at  $O(\alpha_s)$ ) to the average gluon energy,

$$\langle M_H^2 - M_L^2 \rangle = \sqrt{s} \langle E_g \rangle. \quad (1)$$

The more energy is carried away by the gluon, the more asymmetric the hemisphere

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<sup>2</sup>These initial state radiation effects can be incorporated in a standard way [8, 9] and will not be addressed further here.

masses. In practice, as we shall see,  $\langle E_g \rangle$  can be  $O(10 \text{ GeV})$ , which is much larger than the estimated statistical errors on  $M$  from such analyses. A Monte Carlo which does not take full account of gluon radiation at the  $t\bar{t}$  production stage could yield misleading results on the precision on  $M$ .

Another method which is more suited to a heavier top quark is to reconstruct the mass directly from the three-jet final state [12]. To avoid combinatorial and other backgrounds, the energy of the three-jet system is typically constrained to be close to the beam energy. But once again, the emission before the top decay of an energetic gluon can distort this constraint, since naively

$$\langle E_{3 \text{ jet}} \rangle - E_{\text{beam}} = -\frac{1}{2}\langle E_g \rangle. \quad (2)$$

Once again we see the importance of taking the effects of gluon radiation into account in reconstructing the top mass.

In this paper we present results for gluon bremsstrahlung in  $e^+e^- \rightarrow t\bar{t}$  which are of practical interest for physics studies of future linear  $e^+e^-$  colliders. We focus in particular on the gluon energy spectrum which, as we have seen, can have important implications for the top mass measurement. It is also straightforward, at least in principle, to extend our results to the more complicated case of hadronic  $t\bar{t}$  production.

Since centre-of-mass energies  $\sqrt{s}$  at which multiple gluon radiation should be taken into account [2, 13] ( $\log(\sqrt{s}/M) \gg 1$ ) appear unrealistic for the foreseeable future, we need only consider a first-order perturbative analysis. This is calculationally quite straightforward, and in fact many parts of the calculation can be taken over from existing QED analyses [14, 15] (see also [16]).

The remainder of the paper is organised as follows. We first set up the theoretical framework which allows us to calculate the gluon radiation distribution for arbitrary production currents. We derive compact, analytical expressions for the gluon energy spectrum and its average value for vector, axial vector, scalar and pseudoscalar production currents. We then present some numerical calculations which illustrate the size of the effects for typical experimental parameters. Finally, we summarize our results and present our conclusions.

## 2 Calculation of QCD bremsstrahlung distributions

In this section we describe the calculation of the cross section for single primary gluon emission in the process  $e^+e^- \rightarrow t\bar{t}$ . Our calculation is valid for any collision energy  $\sqrt{s} > 2M$ , and we take both photon and  $Z$  exchanges into account.

We begin by defining some variables:  $\sqrt{q^2} \equiv \sqrt{s} = 2E$  is the total centre-of-mass energy and  $M$  is the mass of the top quark. The momenta of the initial and final state particles are labelled by  $e^-(k_1) + e^+(k_2) \rightarrow t(p_1) + \bar{t}(p_2) + g(k)$  and we define

$q^\mu = p_1^\mu + p_2^\mu + k^\mu$ . The energy fractions shared by the  $t$ ,  $\bar{t}$  and gluon in the  $e^+e^-$  centre-of-mass frame are

$$\begin{aligned} z_i &= \frac{2q \cdot p_i}{q^2}, & i = 1, 2 \\ z &= \frac{2q \cdot k}{q^2}, & z = 2 - z_1 - z_2. \end{aligned} \quad (3)$$

The quark velocity in the  $t\bar{t}$  centre-of-mass frame is

$$\beta = \beta(z) \equiv \sqrt{1 - \rho_0} = \sqrt{1 - \frac{4\gamma}{1 - z}} \leq v = \sqrt{1 - 4\gamma}, \quad (4)$$

where

$$\gamma = \frac{M^2}{q^2} \leq \frac{1}{4}, \quad \rho_0 = \frac{4\gamma}{1 - z}. \quad (5)$$

The final-state radiation is limited by the maximum kinematically allowed energy

$$0 \leq z \leq z_{\max} = v^2. \quad (6)$$

The virtualities of the top quark and antiquark before gluon emission,  $\kappa_i$ , can be written in terms of the energy fractions:

$$\begin{aligned} \kappa_1 &= 2k \cdot p_1 = q^2(1 - z_2), \\ \kappa_2 &= 2k \cdot p_2 = q^2(1 - z_1). \end{aligned} \quad (7)$$

It is also convenient to introduce the “angular” variable

$$\rho = \frac{\kappa_1}{zq^2} = \frac{1 - z_2}{z}. \quad (8)$$

In the calculations which follow, we neglect the electron mass  $m_e$ . Since we are concerned with energy losses caused by gluon emission at the  $t\bar{t}$  production stage, we may treat the  $t$ -quarks as stable objects. Only very soft gluons with energy  $E_g \lesssim \Gamma_t E/M$  are affected by the top width, and these make a negligible contribution to the average energy loss.<sup>3</sup>

## 2.1 Lowest-order cross section

The total non-radiative cross section for  $t\bar{t}$  production in  $e^+e^-$  collisions can be expressed in terms of the vector and axial contributions (see for example [18, 19]):

$$\sigma_{t\bar{t}} = \sigma^{VV}(s) + \sigma^{AA}(s) = R_{t\bar{t}}(s)\sigma_{\text{pt}}(s), \quad (9)$$

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<sup>3</sup> A detailed discussion of the potential impact of the top quark instability on gluon emission can be found in Refs. [2, 3, 4, 17].

where

$$\sigma_{\text{pt}}(s) = \frac{4\pi\alpha^2(s)}{3s} \quad (10)$$

is the point cross section for  $e^+e^- \rightarrow \mu^+\mu^-$  with the QED coupling evaluated at the scale  $s$ , and

$$R_{t\bar{t}}(s) = R^{VV}(s) + R^{AA}(s) = v (\zeta_V \tau^V(s) + \zeta_A \tau^A(s)) \quad (11)$$

with

$$\zeta_V = \frac{1}{2}(3 - v^2) = 1 + 2\gamma, \quad (12)$$

$$\zeta_A = v^2 = 1 - 4\gamma. \quad (13)$$

The electroweak factors  $\tau^C$  ( $C = A, V$ ) are defined as

$$\begin{aligned} \tau^V &= \frac{4}{3} - 4v_e v_t \kappa \frac{s(s - M_Z^2)}{(s - M_Z^2)^2 + (s\Gamma_Z/M_Z)^2} \\ &\quad + 3(v_e^2 + a_e^2)v_t^2 \kappa^2 \frac{s^2}{(s - M_Z^2)^2 + (s\Gamma_Z/M_Z)^2}, \end{aligned} \quad (14)$$

$$\tau^A = 3(v_e^2 + a_e^2)a_t^2 \kappa^2 \frac{s^2}{(s - M_Z^2)^2 + (s\Gamma_Z/M_Z)^2}, \quad (15)$$

with

$$\begin{aligned} \kappa &= \frac{\sqrt{2}G_F M_Z^2}{4\pi\alpha(s)}, \\ v_e &= -\frac{1}{2} + 2\sin^2\theta_W, \quad a_e = -\frac{1}{2}, \\ v_t &= \frac{1}{2} - \frac{4}{3}\sin^2\theta_W, \quad a_t = \frac{1}{2}. \end{aligned} \quad (16)$$

The functions  $\tau^V(s)$  and  $\tau^A(s)$  are shown in Fig. 1. Except for energies close to the  $Z^0$  peak, which are not relevant for  $t\bar{t}$  production, the axial current piece is numerically small compared to the vector current piece. At very high energies the ratio of  $\tau^A$  to  $\tau^V$  tends to a constant value of approximately 0.26. Furthermore, close to  $t\bar{t}$  threshold the axial current contribution is further suppressed by two powers of  $v$ , see Eqs. (12,13).

We should also note that near threshold, the  $t\bar{t}$  cross section is strongly modified by Coulomb enhancement effects, see Refs. [20, 21, 22]. However the extreme threshold region is not of primary interest in the present study, since the perturbation radiation is strongly suppressed there.

## 2.2 Dalitz plot distribution for the radiative process

We consider in this section the calculation of the differential cross section for the emission of a primary gluon of momentum  $k$  in  $t\bar{t}$  production. After integrating over

the relative orientation of the quark and gluon momenta with respect to the incoming leptons, the interference between the vector and axial pieces vanishes and the double differential cross section in the quark and antiquark energy fractions (the Dalitz plot distribution) can be written as

$$\frac{d^2\mathcal{W}}{dz_1 dz_2} = \frac{1}{\sigma_{t\bar{t}}} \frac{d^2\sigma_g}{dz_1 dz_2} = \frac{R^{VV}}{R_{t\bar{t}}} \left\{ \frac{d^2\mathcal{W}}{dz_1 dz_2} \right\}_V + \frac{R^{AA}}{R_{t\bar{t}}} \left\{ \frac{d^2\mathcal{W}}{dz_1 dz_2} \right\}_A, \quad (17)$$

with

$$\left\{ \frac{d^2\mathcal{W}}{dz_1 dz_2} \right\}_C = \frac{1}{\sigma^{CC}} \frac{d^2\sigma_g^C}{dz_1 dz_2}. \quad (18)$$

The subscript  $C$  ( $= V, A$ ) denotes the production current, see Eqs. (9,11).

The derivation of the expressions for the radiative matrix element squared, summed over colours and spins and integrated over the azimuthal orientation of the produced particles with respect to the beam direction, drastically simplifies if one uses an analogue of the “invariant-integration technique” which was employed in Refs. [14, 15] for calculations of the QED radiation accompanying two-charged-particle production in  $e^+e^-$  annihilation.

The matrix element  $M_C^{(1)}$  describing the radiation off the top quark produced via the  $C$ -current exchange can be written as

$$M_C^{(1)} = g_s B_C T^a \frac{g_{\mu\nu}}{q^2} \left\{ \bar{u}(p_1) A_\mu^C v(p_2) \right\} \cdot J_\nu^{e,C}, \quad (19)$$

with

$$A_\mu^C = \Gamma_\mu^C (j \cdot e_\lambda) + \frac{\hat{e}_\lambda \hat{k} \Gamma_\mu^C}{\kappa_1} + \frac{\Gamma_\mu^C \hat{e}_\lambda \hat{k}}{\kappa_2}. \quad (20)$$

Here  $T^a$  is a SU(3) colour matrix and  $e_\lambda$  is a polarization vector for the gluon. The matrices  $\Gamma_\mu^C$  ( $C = V, A$ ) are

$$\Gamma_\mu^V = \gamma_\mu, \quad \Gamma_\mu^A = \gamma_\mu \gamma_5. \quad (21)$$

The four-vector  $j_\mu$  is the classical current induced by the acceleration of the top-quark colour charges,

$$j_\mu = \frac{2p_{1\mu}}{\kappa_1} - \frac{2p_{2\mu}}{\kappa_2}, \quad (22)$$

and  $J_\mu^{e,C}$  describes the  $C$ -exchange contribution at the electron vertex,

$$J_\mu^{e,C} = \bar{v}(k_2) \Gamma_\mu^C u(k_1). \quad (23)$$

The normalization factors  $B_C$  are chosen so that the non-radiative matrix element  $M_C^{(0)}$  is

$$M_C^{(0)} = B_C \frac{g_{\mu\nu}}{q^2} \left\{ \bar{u}(p_1) \Gamma_\mu^C v(p_2) \right\} \cdot J_\nu^{e,C}. \quad (24)$$

Following Refs. [14, 15], it is convenient to integrate out the angular variables which describe the orientation of the final state quarks and gluons with respect to the incoming leptons. We first introduce the Lorentz-invariant polarization tensor  $\Pi_{\mu\nu}^C$

$$\Pi_{\mu\nu}^C = \frac{1}{16} \sum_{\lambda} \text{Tr} \left[ (\hat{p}_1 + M) A_{\mu}^C (M - \hat{p}_2) \bar{A}_{\nu}^C \right] , \quad (25)$$

with  $\bar{A}_{\nu}^C = \gamma^0 (A_{\nu}^C)^{\dagger} \gamma_0$ , which describes the radiative decay of a polarized spin-one heavy “object”  $C$ . Since we wish to integrate over the directions of the quark and gluon momenta, the resulting tensor structure can depend only on the four-vector  $n_{\nu} = q_{\nu}/\sqrt{q^2}$  which defines the reference frame in which the quark energies are fixed. It is convenient to write the polarization tensor  $\Pi_{\mu\nu}^C$  in the general form

$$\Pi_{\mu\nu}^C = \frac{1}{3} (g_{\mu\nu} - n_{\mu} n_{\nu}) \Pi_C + n_{\mu} n_{\nu} \Pi_C^q , \quad (26)$$

where

$$\Pi_C = \Pi_{\mu\mu} - \Pi_C^q , \quad \Pi_C^q = n_{\mu} \Pi_{\mu\nu}^C n_{\nu} . \quad (27)$$

Note that because of vector current conservation, the tensor  $\Pi_{\mu\nu}^V$  is transverse, i.e.

$$\Pi_V^q = 0 . \quad (28)$$

For the axial current, the second (longitudinal) term in Eq. (26) vanishes in the limit  $m_e \rightarrow 0$ . In what follows we will set  $m_e = 0$ , and so only the transverse component  $\Pi_C$  appears in the final expression for the double-differential distribution, Eq. (18). As a result, we obtain

$$\left\{ \frac{d^2 \mathcal{W}}{dz_1 dz_2} \right\}_C = \frac{1}{v} \frac{C_F \alpha_s}{\pi} \Sigma_C , \quad (29)$$

where

$$\Sigma_C \equiv \frac{1}{4} \frac{\Pi_C}{\Pi_C^{(0)}}$$

with  $\Pi_C^{(0)}$  the non-radiative polarization operator that one obtains from Eqs. (26,27) by substituting the Born vertices for  $A^C$  in Eq. (25),

$$\Pi_{\mu\nu}^{(0)C} = \frac{1}{16q^2} \sum_{\lambda} \text{Tr} \left[ (\hat{p}_1 + M) \Gamma_{\mu}^C (M - \hat{p}_2) \bar{\Gamma}_{\nu}^C \right] .$$

With the numerical normalization chosen for Eq. (25) one has

$$4 \Pi_C^{(0)} = \zeta_C , \quad \Sigma_C = \frac{1}{\zeta_C} \Pi_C , \quad (30)$$

where the Born factors  $\zeta_C$  are given in Eqs. (12,13). Explicit calculations give<sup>4</sup>

$$\Pi_V = j^2 \frac{q^2}{4} \zeta_V + \frac{1}{2} \left( \frac{\kappa_2}{\kappa_1} + \frac{\kappa_1}{\kappa_2} \right), \quad (31)$$

$$\Pi_A = j^2 \frac{q^2}{4} \zeta_A + \frac{1}{2} \left[ \left( \frac{\kappa_2}{\kappa_1} + \frac{\kappa_1}{\kappa_2} \right) (1 + 2\gamma) + 4\gamma \right], \quad (32)$$

with

$$j^2 \equiv -g^{\mu\nu} j_\mu j_\nu = \frac{4}{\kappa_1 \kappa_2} \left[ (q^2 - \kappa_1 - \kappa_2 - 2M^2) - M^2 \left( \frac{\kappa_2}{\kappa_1} + \frac{\kappa_1}{\kappa_2} \right) \right]. \quad (33)$$

In terms of the dimensionless energy fractions defined in Eq. (3), we have

$$\begin{aligned} \frac{q^2}{4} j^2 &= \frac{z_1 + z_2 - 1 - 2\gamma}{(1 - z_1)(1 - z_2)} - \frac{\gamma}{(1 - z_1)^2} - \frac{\gamma}{(1 - z_2)^2}, \\ \frac{\kappa_2}{\kappa_1} + \frac{\kappa_1}{\kappa_2} &= \frac{z^2}{(1 - z_1)(1 - z_2)} - 2. \end{aligned} \quad (34)$$

Finally, the Dalitz plot distribution is given by substituting the expressions given in Eqs. (11,12,13,29-34) into Eqs. (17,18).

Before presenting numerical results, we make some general comments concerning the global structure of the above results for the  $\Pi_C$ .

- (i) The first (“classical”) term in Eqs. (31,32) corresponds to long-distance radiation with polarization vector  $\vec{e}_\parallel$  in the plane defined by  $\vec{p}_1, \vec{p}_2$ . It leads to a universal contribution to the radiative cross section. In contrast, the short-distance effects are represented by the second term, which induces a current-dependent, non-universal contribution. This describes equal production of the gluon states  $\vec{e}_\parallel$  and  $\vec{e}_\perp$  (polarization transverse to the  $\vec{p}_1, \vec{p}_2$  plane).
- (ii) Next, consider the case of soft gluon emission, when the radiative cross section can be expanded in powers of the gluon energy fraction  $z \ll 1$ . In accordance with the factorization properties of soft radiation [23], the first term in Eqs. (31,32) is proportional to the product of the Born term  $\zeta_C$  and the usual accompanying radiation factor  $j^2$ , with the former evaluated at the total centre-of-mass energy  $q^2$  of the process. It is interesting that at small  $z$  the second (non-universal) term in Eqs. (31,32) appears to be  $O(z^2)$  compared to the first term, not  $O(z)$  as one might naively expect. The only correction linear in  $z$  is incorporated into the classical radiation factor  $j^2$ . The physical origin of this can be understood [24] by recalling the celebrated Low soft bremsstrahlung theorem [25], extended to the case of charged fermions by Burnett and Kroll [26]. The

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<sup>4</sup>For completeness, we present in the Appendix analogous formulae for scalar ( $C = S$ ) and pseudoscalar ( $C = P$ ) exchanges.



application of these classical results is especially transparent for the case of the integrated annihilation cross sections, where the non-radiative process depends only on one energy variable (the centre-of-mass energy of the charged particles).<sup>5</sup>

- (iii) In the extreme ultra-relativistic limit,  $\gamma \ll 1$ , the  $\Pi_V$  and  $\Pi_A$  distributions become identical.
- (iv) The results (29-32) are not valid in the extreme threshold limit, even within the context of perturbation theory. In this region, the first-order radiation amplitude is strongly modified by the QCD Coulomb-like interaction between the  $t$  and  $\bar{t}$  [21]. This leads to a distortion of the radiation pattern and to the additional suppression of radiation [21, 28].

## 2.3 Gluon energy spectrum

To calculate the inclusive gluon energy distribution it is convenient to rewrite the quantities  $\Sigma_C$  in terms of the variables  $z$  and  $\rho$  (see Eqs. (8,30-32)). This gives

$$\Sigma_V = \Sigma_{\text{soft}} + \frac{1}{2}z^2 \frac{1}{\zeta_V} \left[ \frac{1}{\rho} - 1 \right] + (1 \leftrightarrow 2) , \quad (35)$$

$$\Sigma_A = \Sigma_{\text{soft}} + \frac{1}{2}z^2 \frac{1}{\zeta_A} \left[ \frac{1}{\rho} - 1 + \frac{2\gamma}{\rho} \right] + (1 \leftrightarrow 2) . \quad (36)$$

Here we have introduced  $\Sigma_{\text{soft}}$  as the classical current contribution which accounts for the  $z^0$  and  $z^1$  terms,

$$\Sigma_{\text{soft}} = \frac{1 - z - 2\gamma}{\rho} - \frac{\gamma}{\rho^2} , \quad (37)$$

and

$$\frac{1}{2}(1 - \beta(z)) \leq \rho \leq \frac{1}{2}(1 + \beta(z)) . \quad (38)$$

Explicit integration then gives

$$z \left\{ \frac{d\mathcal{W}}{dz} \right\}_C = \frac{z}{\sigma^{CC}} \frac{d^2\sigma_g^C}{dz} = \frac{\beta(z)}{v} \frac{C_F\alpha_s}{\pi} \left[ \xi(z) + \zeta_C^{-1} z^2 \xi_H(z; C) \right] , \quad (39)$$

with the Born factors  $\zeta_C$  given in Eqs. (12,13). The universal “soft” part of the gluon density reads

$$\xi(z) = (1 + v^2 - 2z)\mathcal{L}(z) - 2(1 - z) , \quad (40)$$

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<sup>5</sup>The Low-Kroll-Burnett approach leads to simple results also for initial-state radiation in annihilation processes. For example, in  $e^+e^-$  annihilation, the universal part of the integrated photon-emission cross section is given by the product of the Born term taken at the point  $(q - k)^2$  and the radiation factor describing the emission off the initial electrons [14, 15, 24]. This construction remains valid for the more general case of arbitrarily polarized, massive initial-state particles (see also [27]).

where

$$\mathcal{L}(z) = \frac{1}{\beta(z)} \ln \frac{1 + \beta(z)}{1 - \beta(z)} . \quad (41)$$

The “hard” contributions which depend on the the production channel are

$$\begin{aligned} \xi_H(z; V) &= \mathcal{L}(z) - 1 , \\ \xi_H(z; A) &= \mathcal{L}(z) \left[ \frac{3 - v^2}{2} \right] - 1 . \end{aligned} \quad (42)$$

Note that the result for  $\left\{ \frac{d\mathcal{W}}{dz} \right\}_V$  coincides with that obtained in Ref. [14].

## 2.4 Energy integrals

We turn now to the quantity which is the main concern of the present study – the average energy lost by the  $t$ -quark by gluon emission at the production stage.

The average gluon energy fraction is

$$\langle z \rangle_C = \int_0^{v^2} dz \, z \left\{ \frac{d\mathcal{W}}{dz} \right\}_C \quad (43)$$

and the mean energy fraction lost by the  $t$ -quark (antiquark) is then

$$\langle z_1 \rangle = \langle z_2 \rangle = 1 - \frac{1}{2} \langle z \rangle_C . \quad (44)$$

It is convenient to write  $\langle z \rangle_C$  in the form

$$\langle z \rangle_C = C_F \frac{\alpha_s}{\pi} \left\{ \mathcal{Z} + \zeta_C^{-1} \mathcal{Z}_C \right\} , \quad (45)$$

where the universal  $C$ -independent piece is

$$\mathcal{Z} = \frac{2}{v} \int_0^{v^2} dz \, \beta(z) [(1 - z - 2\gamma)\mathcal{L}(z) - (1 - z)] . \quad (46)$$

The “hard” term  $\mathcal{Z}_C$  in Eq. (45) is given by

$$\begin{aligned} \mathcal{Z}_V &= I_{\mathcal{L}} - I_1 , \\ \mathcal{Z}_A &= I_{\mathcal{L}} \left[ \frac{3 - v^2}{2} \right] - I_1 , \end{aligned} \quad (47)$$

with

$$\begin{aligned} I_{\mathcal{L}} &= \frac{1}{v} \int_0^{v^2} dz \, z^2 \beta(z) \mathcal{L}(z) = \frac{1}{v} \int_0^{v^2} dz \, z^2 \ln \frac{1 + \beta(z)}{1 - \beta(z)} , \\ I_1 &= \frac{1}{v} \int_0^{v^2} dz \, z^2 \beta(z) . \end{aligned} \quad (48)$$

Performing the  $z$  integration in Eqs. (46,48), we obtain

$$\mathcal{Z} = \frac{1}{8} \left\{ (3 + 2v^2 + 3v^4) \mathcal{L}_v - 6(1 + v^2) \right\} , \quad (49)$$

$$I_{\mathcal{L}} = \frac{1}{16} \left\{ \left( \frac{5}{3} + v^2 + v^4 + \frac{5v^6}{3} \right) \mathcal{L}_v - 2 \left( \frac{5}{3} + \frac{14v^2}{9} + \frac{5v^4}{3} \right) \right\} , \quad (50)$$

$$I_1 = \frac{1}{16} \left\{ (-5 + 3v^2 + v^4 + v^6) \mathcal{L}_v + 2 \left( 5 - \frac{4v^2}{3} - v^4 \right) \right\} , \quad (51)$$

with

$$\mathcal{L}_v \equiv \frac{1}{v} \ln \frac{1+v}{1-v} . \quad (52)$$

The average gluon energy fractions for vector and axial production currents are obtained by combining the results of Eqs. (43-52). Figure 2 shows the resulting  $\langle z \rangle_V$  and  $\langle z \rangle_A$  as a function of the top quark velocity  $v$ , for  $\alpha_s = 0.1$ . Note that in the relativistic limit,  $v \rightarrow 1$ , the average gluon energies diverge logarithmically,

$$\langle z \rangle_V \approx \langle z \rangle_A \approx C_F \frac{\alpha_s}{\pi} \left[ \frac{4}{3} \ln \frac{2}{1-v} - \frac{22}{9} \right] , \quad (53)$$

reflecting the emergence of the collinear singularity in the massless quark limit. Of course when  $\alpha_s \ln(1/(1-v)) = O(1)$  multiple gluon emission becomes important, and the leading logarithms must be resummed to all orders (see Ref. [2]). The behaviour of the average energy in the threshold region,  $v \rightarrow 0$ , is discussed in the following section.

Figure 3 shows that average gluon energy  $\langle E_g \rangle = \langle z \rangle \sqrt{s}/2$  as a function of the top mass  $M$  in  $e^+e^-$  annihilation at centre-of-mass energy  $\sqrt{s}$ . The correct proportions of vector and axial current contributions (Fig. 1) are included, and the strong coupling is taken to be  $\alpha_s = 0.1$ . For top masses in the experimentally favoured range 120 – 180 GeV,  $\langle E_g \rangle$  varies from 14 GeV to 4 GeV.

## 2.5 Non-relativistic case

For  $v^2 \ll 1$ , we can take the small- $v$  limit of the results of the previous section to obtain

$$\mathcal{Z} \simeq \frac{16}{15} v^4 \left[ 1 + O(v^2) \right] , \quad (54)$$

$$I_{\mathcal{L}} \simeq \frac{2v^2}{7} \mathcal{Z} , \quad (55)$$

$$I_1 \simeq \frac{v^2}{7} \mathcal{Z} . \quad (56)$$

The high powers of the threshold factor  $v$  originate as follows: in the case of the “classical” universal term  $\mathcal{Z}$ , one power of  $v^2$  comes from the size of the integration region,  $z_{\max} = v^2$  (Eq. (6)), and another power comes from the dipole suppression of the accompanying radiation, see also Refs.[15, 21]. Because of the explicit  $z^2$  factor appearing in the “hard radiation” spectrum, Eq. (39), the non-universal contribution  $\mathcal{Z}_C \sim z_{\max}^3 = v^6$ , and thus acquires an additional  $v^2$  suppression relative to the universal  $\mathcal{Z}$  piece. Note, however, that the impact of the “hard radiation” in the threshold region strongly depends on the production channel. In particular, the Born factor  $\zeta_C$  in the denominator in Eq. (45) can compensate the suppression of the non-classical effects in  $\mathcal{Z}_C$ , see Eq. (12,13). This is an interesting example of how the universal nature of soft radiation can sometimes be obscured by process-dependent short-distance effects.

In the vector channel, where the ( $S$ -wave) Born amplitude is not suppressed at threshold, the soft piece dominates and we find (see also [15, 21])

$$\langle z \rangle_V \sim C_F \frac{\alpha_s}{\pi} \frac{16}{15} v^4 \left[ 1 + O(v^2) \right] . \quad (57)$$

However, in the case of the axial-vector ( $P$ -wave) production channel, the non-radiative amplitude for the  $C \rightarrow q\bar{q}$  decay is additionally suppressed at threshold. As discussed above, “non-classical” radiation effects at  $O(\alpha_s)$  are then equally important, and induce a significant contribution to  $\langle z \rangle_A$ . From Eqs. (45,47,54-56) we find

$$\langle z \rangle_A \sim C_F \frac{\alpha_s}{\pi} \frac{16}{15} v^4 \left( 1 + \frac{2}{7} \right) \left[ 1 + O(v^2) \right] , \quad (58)$$

with the second term in the  $()$  brackets coming from the “hard radiation”. The limiting  $v \rightarrow 0$  behaviours corresponding to the leading  $v^4$  terms in  $\langle z \rangle_V$  and  $\langle z \rangle_A$  are shown as dashed lines in Fig. 2.

As we have already explained, the practical consequences of the differences between the average gluon energy in the vector and axial-vector channels for  $e^+e^- \rightarrow t\bar{t}$  are not large, since the axial contribution is so small in the threshold region, see Eqs. (11,12,13,17). It is, however, worth mentioning that the non-universality of the soft radiation in the threshold region is not restricted to the axial channel. The same situation arises in the case of  $t\bar{t}$  production in the scalar (e.g. Higgs exchange) channel,  $S \rightarrow q\bar{q}$ . In the Appendix we present the corresponding results for  $\langle z \rangle_S$ . We find that in the limit  $v \rightarrow 0$  we have  $\langle z \rangle_S \simeq \langle z \rangle_A$ , so that once again the soft radiation contribution is accompanied by a hard radiation term of the same order.

Another example which could be of practical interest concerns gluon radiation accompanying two scalar quark (e.g. stop) production:

$$e^+e^- \rightarrow \tilde{q}\tilde{q} . \quad (59)$$

This is a  $P$ -wave process, dominated by photon exchange.<sup>6</sup> Near threshold, the average energy loss is again given by Eq. (58), illustrating the non-universality of the soft radiation description for this process also.

All the results derived above have assumed that the top is a *stable* particle. However in practice and particularly near threshold, one must be careful to take account of the effects of the top decay width and of non-perturbative fragmentation. Although the former can affect the distribution of soft gluons (i.e. gluons with energy  $E_g \lesssim \Gamma_t$ ) [2, 3], it is not important for the average top energy loss. We can quantify the possible impact of non-perturbative dynamics by using the string-model results of Ref. [29] for the average amount of energy lost by the top quark before it decays. Near the  $t\bar{t}$  threshold this leads to

$$\langle z \rangle_{\text{non PT}} = \frac{2\kappa\hbar}{\Gamma_t M} v, \quad (60)$$

with the so-called string tension factor  $\kappa \simeq 1 \text{ GeV/fm}$ . Since  $\Gamma_t \sim M^3$  [5], these non-perturbative effects decrease as  $M^{-4}$ . Taking as a specific example the canonical values  $M = 150 \text{ GeV}$  and  $\Gamma_t = 0.8 \text{ GeV}$ , we obtain

$$\langle \Delta z \rangle_{\text{non PT}} \simeq 3.3 \times 10^{-3} v. \quad (61)$$

Comparing Eqs. (57) and (61), one may conclude that the perturbative prediction is only valid for  $v^2 > 0.17$ .

### 3 Conclusions

It is important that gluon emission in top quark production processes is under control. Otherwise, it may provide a serious source of uncertainty in the determination of the top quark parameters from experimental data. In this paper we have studied gluon bremsstrahlung off the top quarks in  $e^+e^- \rightarrow t\bar{t}$ . We have calculated the exact first-order perturbative expressions for the energy fraction distributions of the  $t$ -quarks. These are valid in a wide energy range above the  $t\bar{t}$  threshold.

To examine the impact of gluon emission on top quark distributions we have calculated the average energy loss at the production stage. This infra-red safe quantity characterizes, for example, the difference between the beam energy and the actual energy shared by the produced  $t$ -quarks.

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<sup>6</sup>Detailed QED results for radiation accompanying two charged scalar particle production can be found in Ref. [15].

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## Appendix

In this Appendix we present the expressions corresponding to single primary gluon emission in the process  $e^+e^- \rightarrow q\bar{q}$ , mediated by the exchange of scalar  $S$  or pseudoscalar  $P$  particles (for example, the Higgs bosons of the Standard Model or its supersymmetric extensions). Now the invariant polarization operator bears no Lorentz indices, and we define (cf. Eq. (25))

$$\Pi_C = \frac{1}{8} \sum_{\lambda} \text{Tr} \left[ (\hat{p}_1 + M) A^C (M - \hat{p}_2) \bar{A}^C \right] , \quad (\text{A1})$$

where the normalization has been fixed to preserve the relation (30). Here the scalar functions  $A^C$  are analogous to  $A_\mu^C$  of Eq. (20) but with the vertex operators

$$\Gamma^S = 1 , \quad \Gamma^P = \gamma_5 , \quad (\text{A2})$$

substituted for  $\Gamma_\mu^C$ .

Defining the double-differential distribution analogously to Eqs. (29), (30) with

$$\zeta_S = \zeta_A = v^2 , \quad \zeta_P = 1 , \quad (\text{A3})$$

we find

$$\Pi_S = j^2 \frac{q^2}{4} v^2 + \frac{1}{2} \left( \frac{\kappa_2}{\kappa_1} + \frac{\kappa_1}{\kappa_2} + 2 \right) , \quad (\text{A4})$$

$$\Pi_P = j^2 \frac{q^2}{4} + \frac{1}{2} \left( \frac{\kappa_2}{\kappa_1} + \frac{\kappa_1}{\kappa_2} + 2 \right) . \quad (\text{A5})$$

The gluon energy spectrum is then given by Eq. (39) with

$$\xi_H(z, S) = \xi_H(z; P) = \mathcal{L}(z) , \quad (\text{A6})$$

and the mean gluon energy fraction by Eq. (45) with

$$\mathcal{Z}_S = \mathcal{Z}_P = I_{\mathcal{L}} . \quad (\text{A7})$$

Near the threshold for the pseudoscalar channel, the soft term dominates and

$$\langle z \rangle_P \simeq \langle z \rangle_V . \quad (\text{A8})$$

In contrast, the transition  $O^+ \rightarrow q\bar{q}$  is  $P$ -wave, and the Born term  $\zeta_S = v^2$  compensates the additional suppression of  $\mathcal{Z}_S$ . The threshold result for  $\langle z \rangle_S$  therefore coincides with that for  $\langle z \rangle_A$ , Eqs. (54-56).

## References

- [1] P. Langacker, University of Pennsylvania preprint URP-0555-T (1993).
- [2] Yu.L. Dokshitzer, V.A. Khoze and S.I. Troyan, University of Lund preprint LU-TP-92-10 (1992).
- [3] V.A. Khoze, L.H. Orr and W.J. Stirling, Nucl. Phys. **B378** (1992) 413.  
V.A. Khoze, J. Ohnemus and W.J. Stirling, Phys. Rev. **D49** (1994) 1237.
- [4] Yu.L. Dokshitzer, V.A. Khoze, and S.I. Troyan, in M. Derrick, ed., *Proc. 6th Int. Conf. on Physics in Collision*, World Scientific, 1987, p. 417.
- [5] I.I. Bigi, Yu.L. Dokshitzer, V.A. Khoze, J.H. Kühn and P.M. Zerwas, Phys. Lett. **181B** (1986) 157.
- [6] CDF collaboration: F. Abe *et al.*, Phys. Rev. **D45** (1992) 3921.  
D0 collaboration: S. Abachi *et al.*, Fermilab preprint FERMILAB-PUB-94-004-E (1994).
- [7] J.H. Kühn, Acta Phys. Austr. Suppl. **24** (1982) 203.
- [8] L.N. Lipatov and V.A. Khoze, in Materials of the X Winter School of the Leningrad Nuclear Physics Institute, Vol. II, Leningrad (1975), p. 409.  
V.N. Baier *et al.*, Phys. Rep. **78** (1981) 294.
- [9] V.S. Fadin and E.A. Kuraev, Sov. Jour. Nucl. Phys. **41** (1985) 466.
- [10] Proc. Workshop on  $e^+e^-$  collisions at 500 GeV: the Physics Potential, Munich-AnneCy-Hamburg 1991, ed. P.M. Zerwas, DESY 92-123A,B.
- [11] Proc. Workshop on Physics and Experiments with  $e^+e^-$  Linear Colliders, Saariselkä 1991, eds. R. Orava, P. Eerola and M. Nordberg, World Scientific, Singapore 1992.
- [12] P. Igo-Kemenes *et al.*, Ref. [10], p. 327.
- [13] Yu.L. Dokshitzer, Light and Heavy Quark Jets in Perturbative QCD, in Physics up to 200 TeV, Proc. Intl. School of Subnuclear Physics, Erice 1990, Plenum Press, New York, 1991, Vol. 28.
- [14] V.N. Baier and V.A. Khoze, preprint NSU, Novosibirsk (1964), Sov. Phys. JETP **21** (1965) 629.
- [15] V.N. Baier and V.A. Khoze, Sov. Phys. JETP **21** (1965) 1145.



- [16] B.L. Ioffe, Phys. Lett. **78B** (1978) 277.
- [17] Yu.L. Dokshitzer, V.A. Khoze, L.H. Orr and W.J. Stirling, Nucl. Phys. **B403** (1993) 65.
- [18] J. Jerzak, E. Laermann and P.M. Zerwas, Phys. Rev. **D25** (1982) 363.
- [19] For a recent review see: J.H. Kühn and P.M. Zerwas, Advanced Series on Directions in High Energy Physics, “Heavy Flavours”, eds. A.J. Buras and M. Lindner, World Scientific, Singapore 1992, p. 434.
- [20] V.S. Fadin and V.A. Khoze, JETP Lett. **46** (1987) 525; Sov. J. Nucl. Phys. **48** (1988) 309.
- [21] V.S. Fadin and V.A. Khoze, Proc. 24th LNPI Winter School, Leningrad 1989, Vol. I, p. 3.
- [22] M.J. Strassler and M.E. Peskin, Phys. Rev. **D43** (1991) 1500.
- [23] See for example: J. M. Jauch and F. Rohrlich, *The theory of photons and electrons: the relativistic quantum field theory of charged particles with spin one-half*, (1976); A.I. Akhiezer and V.B. Berestetskii, *Quantum electrodynamics*, Interscience (1965).
- [24] Yu.L. Dokshitzer and V.A. Khoze, in preparation.
- [25] F.E. Low, Phys. Rev. **110** (1958) 974.
- [26] T.H. Burnett and N.M. Kroll, Phys. Rev. Lett. **20** (1968) 86.
- [27] V.N. Baier and V.A. Khoze, Sov. J. Nucl. Phys. **5** (1967) 898.  
I.A. Nagorskaya and V.A. Khoze, Izvestia AN Armenian SSR, Fizika **3** (1968) 371.
- [28] V.S. Fadin and O.I. Yakovlev, Sov. J. Nucl. Phys. **50** (1989) 1037.
- [29] T. Sjöstrand and P.M. Zerwas, Proc. Workshop on  $e^+e^-$  collisions at 500 GeV: the Physics Potential, Munich-Annecy-Hamburg 1991, ed. P.M. Zerwas, DESY 92-123A, p. 463.

## Figure Captions

- [1] The functions  $\tau^V(s)$  and  $\tau^A(s)$  defined in Eqs. (14,15).
- [2] The average gluon energy in  $e^+e^- \rightarrow t\bar{t}g$  as a function of the top quark velocity  $v$ , for vector and axial vector current production. Also shown (dashed lines) are the limiting  $v \rightarrow 0$  behaviours given in Eqs. (57,58), and the limiting  $v \rightarrow 1$  behaviour given in Eq. (53).
- [3] The average gluon energy in  $t\bar{t}$  production in  $e^+e^-$  annihilation at  $\sqrt{s} = 500$  GeV, as a function of the top quark mass  $M$ . The strong coupling is fixed at  $\alpha_s = 0.1$ .

**Fig. 1**

**Fig. 2**

**Fig. 3**

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